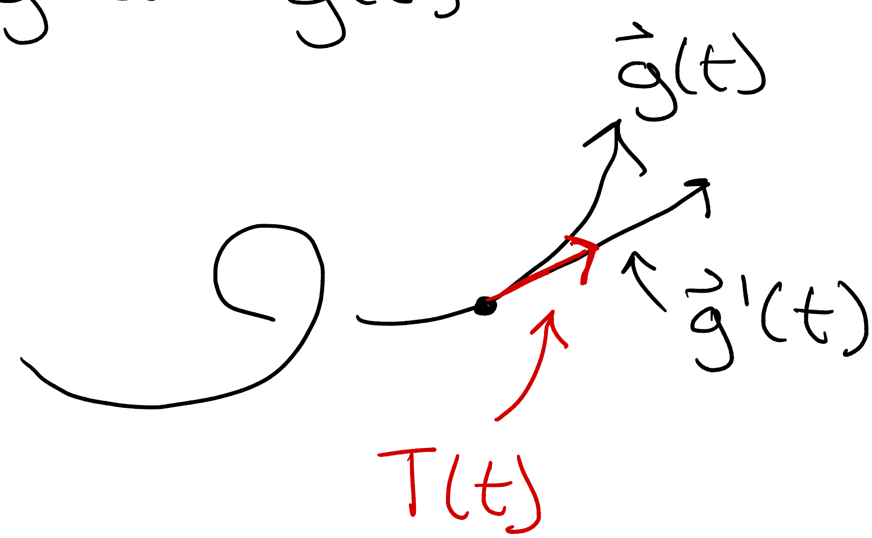
8.3 Line Integrals and Green's Theorem

Definition. A <u>vector field</u> $\vec{F}$ on an open set $U \subset \mathbb{R}^n$ is a function $\vec{F}: U \to \mathbb{R}^n$ which associates a vector to each point in U.

Now we previously (curves, 3.5) defined a parametrized curve to be a map $\vec{g}: [a,b] \to \mathbb{R}^n$. Recall that

$$T(t) = \frac{\vec{g}'(t)}{\|\vec{g}'(t)\|}$$

is called the <u>unit tangent vector</u> to $\vec{g}$ at $\vec{g}(t)$



and that we called a parametrization <u>regular</u> when $\|\vec{g}'(t)\| = 0$ (so T is well-defined). Looking back on this, we now say "a parametrization is regular when rank $Dg = 1$".

Construction. Every 1-form on $U \subset \mathbb{R}^n$ ω has a corresponding vector field $\vec{F}$ so that

$$\omega = \sum F_i \, dx_i \iff \vec{F} = \begin{bmatrix} F_1 \\ \vdots \\ F_n \end{bmatrix}$$

Then if $C \subset \mathbb{R}^n = \vec{g}([a,b])$,

pullback of F_i

$$\int_C \omega = \int_{[a,b]} \vec{g}^* \omega = \int_{[a,b]} \sum_{i=1}^n F_i(\vec{g}(t)) \, g_i'(t) \, dt$$

pullback of dx_i

$$= \int_{[a,b]} \vec{F}(\vec{g}(t)) \cdot \vec{g}\,'(t)\, dt$$

This is called a "line integral" or "path integral".

If $\|\vec{g}\,'(t)\| = 1$ for all $t \in [a,b]$, we say "g is parametrized by arclength" and use s for the parameter. This motivates us to write our path integral as

$$= \int_{[a,b]} \vec{F}(\vec{g}(t)) \cdot \frac{\vec{g}\,'(t)}{\|\vec{g}\,'(t)\|} \|\vec{g}\,'(t)\|\, dt$$

$$= \int_{C} \vec{F} \cdot T\, ds$$

definitions

We've been a bit informal about the smoothness of $\vec{g}$. Everything makes sense as long as C is parametrized by a finite collection of C^1 maps

$$\vec{g}_1 : [a_1, b_1] \to \mathbb{R}^n$$
$$\vdots$$
$$\vec{g}_s : [a_s, b_s] \to \mathbb{R}^n$$

where $\vec{g}_j(b_j) = \vec{g}_{j+1}(a_{j+1})$. We call these curves "piecewise C^1".

Proposition. If C is a curve in $\mathbb{R}^n$ parametrized by $\vec{g}:[a,b] \to \mathbb{R}^n$, let C^- be the curve parametrized by $\vec{h}:[\vec{a},\vec{b}] \to \mathbb{R}^n$, $\vec{h}(u) = \vec{g}(a+b-u)$. Then for every $\omega \in \mathcal{A}^1(\mathbb{R}^n)$, we have

$$\int_C \omega = -\int_{C^-} \omega$$

Proof. We write

$$\int_{C^-} \omega = \int_{[a,b]} \vec{h}^* \omega = \int_a^b F(\vec{h}(u)) \cdot \vec{h}'(u)\, du$$

$$= \int_a^b F(\vec{g}(a+b-u)) \cdot (-\vec{g}'(a+b-u))\, du$$

$$= -\int_a^b F(\vec{g}(t)) \cdot \vec{g}\,'(t)\, dt$$

where $t = a+b-u$

$$= -\int_{[a,b]} \vec{g}^* \omega.$$

$$= -\int_C \omega. \qquad \blacksquare$$

Example. Let C be the line segment $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ to $\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$ and let $\omega = xy\, dz$. To compute $\int_C \omega$, we parametrize C by

$$\vec{g}(t) = (1-t)\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

where $t \in [0,1]$.

We then have

$$\vec{g}'(t) = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}$$

and can compute

$$\int_C \omega = \int_{[0,1]} \vec{g}^* \omega =$$

$$= \int_0^1 \underbrace{((1-t)1 + t \cdot 2)}_{x(\vec{g}(t))} \underbrace{((1-t)(-1) + t \cdot 2)}_{y(\vec{g}(t))} \underbrace{2dt}_{\vec{g}^* dz}$$

$$= \int_0^1 (1+t)(-1+3t)\, 2\, dt$$

$$= \int_0^1 6t^2 + 4t - 2\, dt = 2.$$

Definition. If $\vec{F}(\vec{x}) : \mathbb{R}^n \to \mathbb{R}^n$ is a vector field whose value is a force at each point in space, and ω is the corresponding 1-form, the <u>work</u> done by the force field by a particle moving along a path C is

$$\text{work} = \int_C \omega$$

Definition. If a particle of mass m has velocity vector $\vec{v}$, the kinetic energy $KE = \frac{1}{2} m \|\vec{v}\|^2$.

We can now prove

Work-Energy Theorem. If the only force acting on a particle of mass m causes the particle to move along a path C, then

work = change in kinetic energy.

Proof. Suppose the force is given by $F(\vec{x})$ and the path by $\vec{g}(t)$.

$$\text{work} = \int_C \omega = \int_a^b \vec{F}(\vec{g}(t)) \cdot \vec{g}\,'(t)\, dt$$

$$= \int_a^b m\vec{g}\,''(t) \cdot \vec{g}\,'(t)\, dt$$

$$= \frac{m}{2}\int_a^b \frac{d}{dt}\left(\|\vec{g}\,'(t)\|^2\right) dt$$

$$= \frac{1}{2} m \left(\|\vec{g}\,'(b)\|^2 - \|\vec{g}\,'(a)\|^2 \right)$$

= change in kinetic energy. $\boxtimes$

Now we can prove the fundamental theorem of calculus for line integrals.

Proposition. If $\omega = df \in A^1(\mathbb{R}^n)$ and C is a path from $\vec{a}$ to $\vec{b}$ in $\mathbb{R}^n$,

$$\int_C \omega = f(\vec{b}) - f(\vec{a}).$$

Proof. Suppose C is parametrized by $\vec{g}$.

$$\int_C \omega = \int_a^b \vec{g}^{\,*}\omega = \int_a^b \vec{g}^{\,*}(df)$$

$$= \int_a^b d(\vec{g}^{\,*}f) = \int_a^b d(f \circ \vec{g})$$

$$= \int_a^b (f \circ \vec{g})'(t)\,dt$$

$$= (f \circ \vec{g})(b) - (f \circ \vec{g})(a)$$

$$= f(\vec{g}(b)) - f(\vec{g}(a))$$

$$= f(\vec{b}) - f(\vec{a}). \qquad \square$$

Corollary. If $\vec{F}(\vec{x}) = \nabla f(\vec{x})$, then

$$\int_C \vec{F} \cdot \vec{T}\,ds = \vec{F}(\vec{b}) - \vec{F}(\vec{a}).$$

Notice that the value of the integral doesn't depend on the path!

Definition. If $\omega = d\eta$, we say that η is a <u>potential form</u> for ω.

(or a potential <u>function</u> if $\eta \in \mathcal{A}^0(U)$.)

Theorem. Let $\omega = \sum F_i dx_i \in A^1(U)$ with $U \subset \mathbb{R}^n$. The following are equivalent:

1) For every closed path $C \subset U$,

$$\int_C \omega = 0.$$

2) If $\vec{a}$ and $\vec{b}$ are joined by paths $C \subset U$ and $C' \subset U$,

$$\int_C \omega = \int_{C'} \omega$$

(In this case, we say the integral is path independent and write $\int_{\vec{a}}^{\vec{b}} \omega = \int_C \omega$ for any C which starts at $\vec{a}$ and ends at $\vec{b}$.)

3) $\omega = df$ for some potential function $f: U \to \mathbb{R}$

Note. If a force field $\vec{F} = \nabla f$, we say $\vec{F}$ is <u>conservative</u>. If so, and C is a path parametrized by $\vec{g}$,

$$\int_C \omega = \int_a^b \vec{F}(\vec{g}(t)) \cdot \vec{g}'(t)\,dt$$

$$= \text{work}$$

$$= \frac{1}{2} m \|\vec{g}'(b)\|^2 - \frac{1}{2} m \|\vec{g}'(a)\|^2$$

(by work-energy theorem) but also

$$\int_C \omega = f(\vec{g}(b)) - f(\vec{g}(a))$$

(by fundamental theorem of calculus)

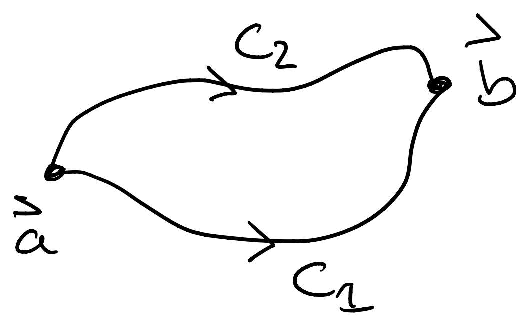
This leads physicists to call $-f(\vec{x})$ a <u>potential energy</u> for $\vec{F}(\vec{x})$

so that they can write above as

$$\Delta K.E. = -\Delta P.E. \Rightarrow \Delta(K.E. + P.E.) = 0$$

and say "the sum of kinetic and potential energy is conserved."

Proof. $(1 \Rightarrow 2)$



If C_1, C_2 are paths from $\vec{a}$ to $\vec{b}$, $C_1 \cup C_2^-$ is a closed path, so

$$0 = \int_{C_1 \cup C_2^-} \omega = \int_{C_1} \omega + \int_{C_2^-} \omega = \int_{C_1} \omega - \int_{C_2} \omega$$

proposition about reparametrization

so $\displaystyle\int_{C_1} \omega = \int_{C_2} \omega.$

$(2 \Rightarrow 3)$ Fix any $\vec{a} \in U$ and define

$$f(\vec{x}) = \int_{\vec{a}}^{\vec{x}} \omega$$

(by hypothesis, the path doesn't matter).

To prove $df = \omega$, we must show

$$F_i(\vec{x}) = \frac{\partial f}{\partial x_i}(\vec{x})$$

$$= \lim_{h \to 0} \frac{f(\vec{x} + h\vec{e}_i) - f(\vec{x})}{h}$$

$$= \lim_{h \to 0} \frac{1}{h} \int_{\vec{x}}^{\vec{x} + h\vec{e}_i} \omega$$

Now we can join $\vec{x}$ to $\vec{x} + h\vec{e}_i$ by

$\vec{g} : [0, h] \to \mathbb{R}^n$, $\vec{g}(t) = \vec{x} + t\vec{e}_i$.

$$= \lim_{h \to 0} \frac{1}{h} \int_0^h \vec{g}^* \omega$$

$$= \lim_{h \to 0} \frac{1}{h} \int_0^h \sum_{j=1}^n F_j(\vec{g}(t)) \vec{g}^* dx_j$$

but

$$\vec{g}^* dx_j = g_j'(t) dt$$

where g_j is the coordinate function.

But $\vec{g}(t) = \vec{x} + t\vec{e}_i$, so

$$g_j'(t) = \delta_{ij}$$

so we have

$$= \lim_{h\to 0} \frac{1}{h} \int_0^h F_i(\vec{g}(t))\, dt$$

$$= \lim_{h\to 0} F_i(\vec{g}(t_*)) \quad \text{for some } t_* \in [0,h] \text{ by mvt for integrals}$$

$$= F_i(\vec{g}(0)) \quad \text{continuity of } F_i$$

$$= F_i(\vec{x}).$$

which proves $df = \omega$, as desired.

(3 => 1) Since $\omega = df$, if C is closed

$$\int_C \omega = \int_C df = f(\vec{a}) - f(\vec{a}) = 0. \quad \blacksquare$$

Definition. If $\omega \in A^k(U)$ and $d\omega = 0$, we say ω is <u>closed</u>. If $\omega = d\eta$ for some $\eta \in A^{k-1}(U)$ we say ω is <u>exact</u>.

Now if $\omega = df$, then $d\omega = d(df) = 0$. So every exact form is closed. Is every closed form exact? The answer will be interesting.

Example. Suppose

$$\omega = (e^x + 2xy)\,dx + (x^2 + \cos y)\,dy$$

We would like to find a potential function f so that $df = \omega$.

Such a function (if it exists) has

$$\frac{\partial f}{\partial x} = e^x + 2xy, \quad \frac{\partial f}{\partial y} = x^2 + \cos y.$$

We can find it by "partial integration"

$$\int e^x + 2xy \, dx = e^x + x^2 y + C(y)$$

partial differentiation

$$\frac{\partial}{\partial y} e^x + x^2 y + C(y) = x^2 + C'(y)$$

solve for $C'(y)$

$$x^2 + \cos y = x^2 + C'(y)$$

$$C'(y) = \cos y$$

integrate again

$$C(y) = \int \cos y \, dy = -\sin y + D$$

assemble results:

$$f(x,y) = e^x + x^2y - \sin y + D.$$

We want to prove a theorem about when this works, but need a tool. Suppose $f : [a,b] \times [c,d] \to \mathbb{R}$ is C^1 and consider

$$F(x) = \int_c^d f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) dy.$$

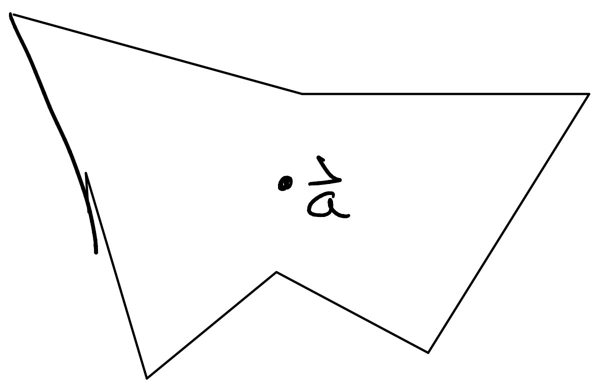
You proved in homework that

$$F'(x) = \frac{\partial}{\partial x} \int_c^d f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) dy$$

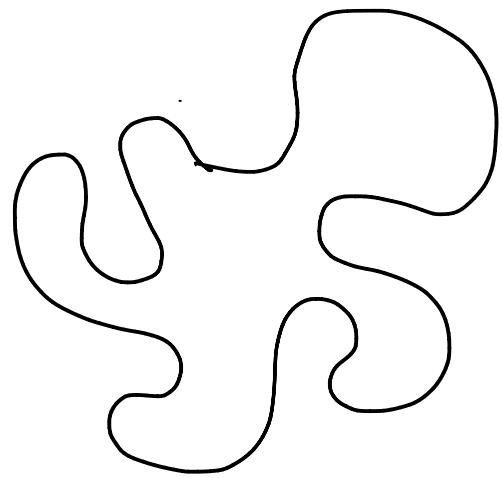
$$= \int_c^d \frac{\partial}{\partial x} f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) dy$$

This is called "differentiating under the integral sign".

Definition. We say a region $\Omega \subset \mathbb{R}^n$ is starlike if there is some $\vec{a} \in \Omega$ so that for every $\vec{x} \in \Omega$, the line segment $\vec{a}\vec{x} \subset \Omega$.



starlike



not starlike

Theorem. Let $\Omega \subset \mathbb{R}^n$ be a starlike region and $\omega \in A^1(\Omega)$. If ω is closed, then ω is exact.

Proof. Suppose $\omega = \sum_i F_i \, dx_i$. For any $\vec{x} \in \Omega$, we can parametrize the line C from $\vec{a}$ to $\vec{x}$ by

$$\vec{g}(t) = \vec{a} + t(\vec{x} - \vec{a}), \quad t \in [0,1].$$

We define

$$f(\vec{x}) = \int_C \omega = \int_{[0,1]} \vec{g}^* \omega$$

$$= \int_0^1 \sum_{j=1}^n F_j(\vec{g}(t))\, g_j'(t)\, dt$$

Now $\vec{g}'(t) = \vec{x} - \vec{a}$, so $g_j'(t) = x_j - a_j$.

$$= \sum_{j=1}^n (x_j - a_j) \int_0^1 F_j(\vec{g}(t))\, dt$$

We claim that $df = \omega$. So we have to compute

$$\frac{\partial f}{\partial x_i} = \int_0^1 F_i(\vec{g}(t))\,dt +$$

$$+ \sum_{j=1}^{n} (x_j - a_j) \frac{\partial}{\partial x_i} \int_0^1 F_j(\vec{g}(t))\,dt$$

$$= \int_0^1 F_i(\vec{g}(t))\,dt +$$

$$+ \sum_{j=1}^{n} (x_j - a_j) \int_0^1 \frac{\partial}{\partial x_i} F_j(\vec{g}(t))\,dt.$$

Now

$$\frac{\partial}{\partial x_i} F_j(\vec{g}(t)) = \frac{\partial}{\partial x_i} F_j(\vec{a} + t(\vec{x} - \vec{a}))$$

$$= \frac{\partial F_j}{\partial x_i}(\vec{a} + t(\vec{x} - \vec{a})) \cdot \frac{\partial}{\partial x_i}(\vec{a} + t(\vec{x} - \vec{a}))$$

$$= \frac{\partial F_j}{\partial x_i}(\vec{g}(t)) \cdot t$$

Now ω is closed, so $d\omega = 0$.

But $$d\omega = \sum_{1\le i<j\le n} \left(\frac{\partial F_i}{\partial x_j} - \frac{\partial F_j}{\partial x_i}\right) dx_i \wedge dx_j,$$

so this means that $\frac{\partial F_i}{\partial x_j} = \frac{\partial F_j}{\partial x_i}$, and we can write

$$\int_0^1 t\,\frac{\partial F_j}{\partial x_i}(\vec{g}(t))\,dt = \int_0^1 t\,\frac{\partial F_i}{\partial x_j}(\vec{g}(t))\,dt$$

and we have

$$\sum_{j=1}^{n} (x_j - a_j) \int_0^1 \frac{\partial}{\partial x_i} F_j(\vec{g}(t))\,dt =$$

$$= \int_0^1 t \underbrace{\sum_{j=1}^{n} (x_j - a_j)\frac{\partial F_i}{\partial x_j}(\vec{g}(t))\,dt}_{\text{the derivative } \frac{d}{dt} F_i(\vec{g}(t)) \text{ by chain rule!}}$$

$$= \int_0^1 t\,(F_i \circ \vec{g})'(t)\,dt$$

$$= t(F_i \circ \vec{g})(t)\Big|_{t=0}^{1} - \int_0^1 F_i(\vec{g}(t))\,dt$$

integration by parts!

$$= F_i(\vec{g}(1)) - \int_0^1 F_i(\vec{g}(t))\,dt$$

$$= F_i(\vec{x}) - \int_0^1 F_i(\vec{g}(t))\,dt$$

Thus

$$\frac{\partial f}{\partial x_i} = \int_0^1 F_i(\vec{g}(t))\,dt + $$

$$+ \sum_{j=1}^{n} (x_j - a_j) \int_0^1 \frac{\partial}{\partial x_i} F_j(\vec{g}(t))\,dt$$

$$= F_i(\vec{x}),$$ as required. $\boxed{}$

This theorem is usually enough: given a 1-form, check if it's closed, then try to construct a potential.

Example. Newton's law of gravitational says that the force of gravity exerted by a point mass M at $\vec{0}$ is given by

$$\vec{F} = -GM \frac{\vec{x}}{\|\vec{x}\|^3}$$

The corresponding 1-form is

$$\omega = \frac{-GM}{(x^2+y^2+z^2)^{3/2}} (x\,dx + y\,dy + z\,dz)$$

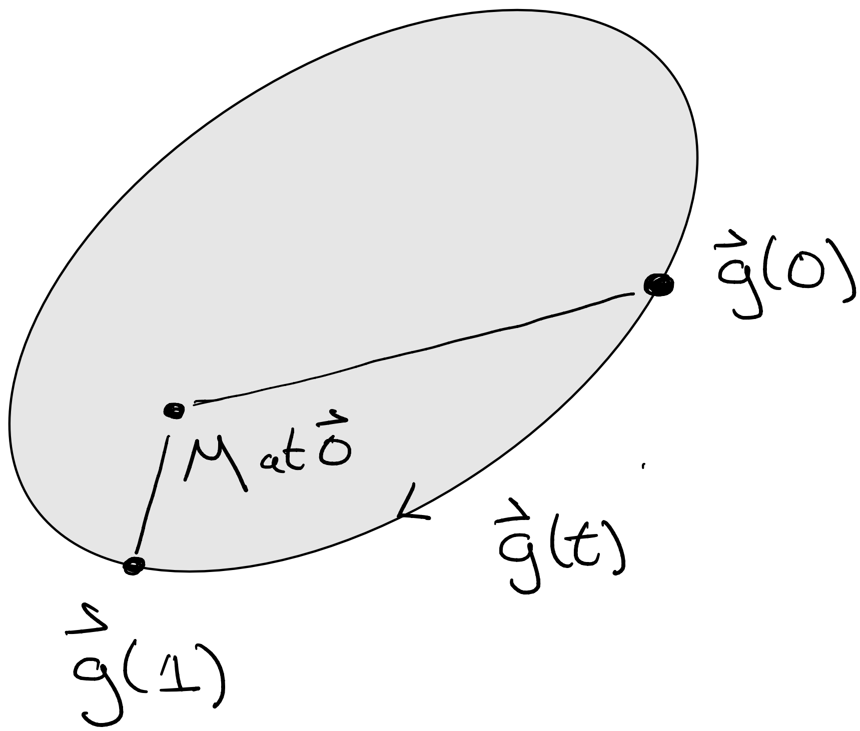
To find a potential, let's try

$$\int \frac{-\cancel{x}}{(x^2+y^2+z^2)^{3/2}} dx = \frac{1}{(x^2+y^2+z^2)^{1/2}} + C$$

and observe that this works, so

$$f(\vec{x}) = \frac{GM}{\|\vec{x}\|}$$ is a potential function.

This means that



$$\text{work} = \int_0^1 \vec{g}^* \omega$$

$$= f(\vec{g}(1)) - f(\vec{g}(0))$$

$$= GM\left(\frac{1}{\|\vec{g}(1)\|} - \frac{1}{\|\vec{g}(0)\|}\right)$$

= change in Kinetic energy

$$= \frac{1}{2} \|\vec{g}'(1)\|^2 - \frac{1}{2} \|\vec{g}'(0)\|^2.$$

and we can see that an object in orbit is moving fastest when closest to the origin.

We can also see that $\|\vec{g}'(t)\|$ is periodic - over a complete orbit no work is done, so the starting and ending Kinetic energy as the same.

Green's Theorem on a Rectangle.

We have proved that if $\omega = df$, and C is a curve from $\vec{a}$ to $\vec{b}$, we have $\int_C \omega = f(\vec{b}) - f(\vec{a})$. This is a 1-d generalization of the fundamental theorem of calculus. Let's try for 2d!

Theorem. (Green's theorem) Let $R \subset \mathbb{R}^2$ be a rectangle and let ω be a 1-form on R. Then if ∂R is the closed curve given by following the boundary ccw,

$$\int_{\partial R} \omega = \int_R d\omega.$$

Proof. Suppose $R = [a,b] \times [c,d]$ and

$$\omega = P\,dx + Q\,dy$$

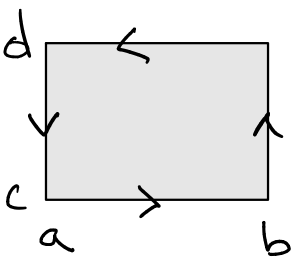
Now $d\omega = \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx \wedge dy$, and

$$\int_R d\omega = \int_R \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) d\text{Area}$$

$$= \int_c^d \left(\int_a^b \frac{\partial Q}{\partial x}\,dx\right) dy - \int_a^b \left(\int_c^d \frac{\partial P}{\partial y}\,dy\right) dx$$

$$= \int_c^d Q\left(\begin{bmatrix} b \\ y \end{bmatrix}\right) - Q\left(\begin{bmatrix} a \\ y \end{bmatrix}\right) dy$$

$$- \int_a^b P\left(\begin{bmatrix} x \\ d \end{bmatrix}\right) - P\left(\begin{bmatrix} x \\ c \end{bmatrix}\right) dx$$

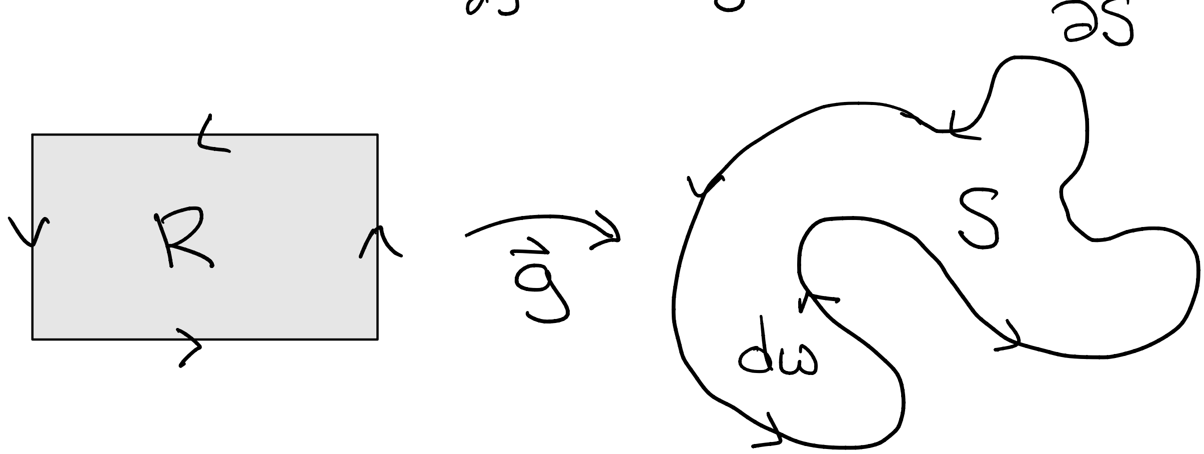


$$= \int_a^b P\left(\begin{bmatrix} x \\ c \end{bmatrix}\right) dx + \int_c^d Q\left(\begin{bmatrix} b \\ y \end{bmatrix}\right) dy$$

$$+ \int_b^a P\left(\begin{bmatrix} x \\ d \end{bmatrix}\right) dx + \int_d^c Q\left(\begin{bmatrix} a \\ y \end{bmatrix}\right) dy$$

$$= \int_{\partial R} \omega. \qquad \blacksquare$$

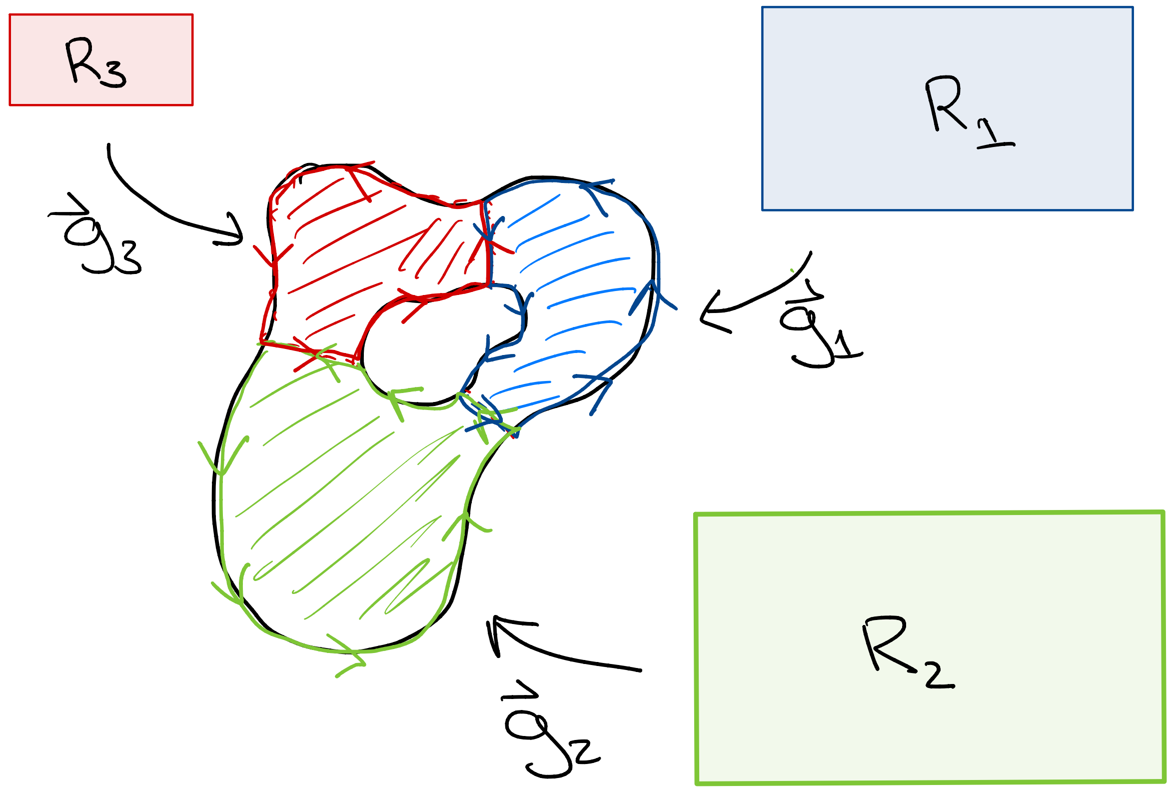
<u>Corollary</u>. If $S \subset \mathbb{R}^2$ is parametrized by a rectangle and ω is a 1-form on S, then $\int_{\partial S} \omega = \int_S d\omega$.



Proof.

$$\int_{\partial S} \omega = \int_{\partial R} \vec{g}^* \omega = \int_R d(\vec{g}^* \omega) = \int_R \vec{g}^* (d\omega) = \int_S d\omega.$$

We observe that if we can "tile" a region by subsets parametrized by rectangles, the theorem works.



We note that sections of the boundary of the rectangles which map to the interior of S cancel each other out, leaving only ∂S.

Example. Suppose that $\omega = \frac{-y}{2}dx + \frac{x}{2}dy$.
Then $d\omega = -\frac{1}{2}dy\wedge dx + \frac{1}{2}dx\wedge dy = dx\wedge dy$.
So

$$\text{area}(S) = \int_S dx\wedge dy = \int_S d\omega = \int_{\partial S} \omega.$$

and so we can compute the area enclosed by any curve $C \subset \mathbb{R}^2$ by integrating

$$\frac{1}{2}\int_C -y\,dx + x\,dy = \text{area enclosed!}$$

Now we can say a bit more about closed and exact forms.

Definition. A subset $X \subset \mathbb{R}^n$ is called <u>path connected</u> if every $\vec{a}, \vec{b} \in X$ are the endpoints of a curve $C \subset X$.

Definition. A subset $X \subset \mathbb{R}^n$ is called <u>simply connected</u> if it is path connected and every closed curve $C \subset X$ may be parametrized by $\vec{g} : R \subset \mathbb{R}^2 \to X$ so that $C = \vec{g}(\partial R)$.

Corollary. Let $\Omega \subset \mathbb{R}^n$ be a simply connected region, and ω a 1-form on Ω. If ω is closed, ω is exact.

Proof. Suppose C is a closed curve in X. Then $C = \vec{g}(\partial R)$, so by hypothesis $C = \partial S$ where S is parametrized by R. We then have

$$\int_C \omega = \int_{\partial S} \omega \underset{\text{Green's thm corollary}}{=} \int_S d\omega \underset{\omega \text{ is closed}}{=} \int_S 0 = 0.$$

By our previous theorem, if $\int_C \omega = 0$ for every closed curve in X, $\omega = df$. ☒

We now reconsider the example

$$\omega = \frac{-y}{x^2+y^2}\,dy + \frac{x}{x^2+y^2}\,dx$$

We have previously computed

$$d\omega = \left(\frac{\partial}{\partial y} \frac{y}{x^2+y^2} + \frac{\partial}{\partial x} \frac{x}{x^2+y^2} \right) dx \wedge dy$$

$$= \left(\frac{(x^2+y^2) - 2y^2}{(x^2+y^2)^2} + \frac{(x^2+y^2) - 2x^2}{(x^2+y^2)^2} \right) dx \wedge dy$$

$$= 0.$$

Now if $\vec{g}(t) = \begin{bmatrix} r\cos t \\ r\sin t \end{bmatrix}$, we have

$$\vec{g}^{*}\omega = \frac{-r\sin t}{r^2\cos^2 t + r^2\sin^2 t}(-r\sin t)\,dt$$

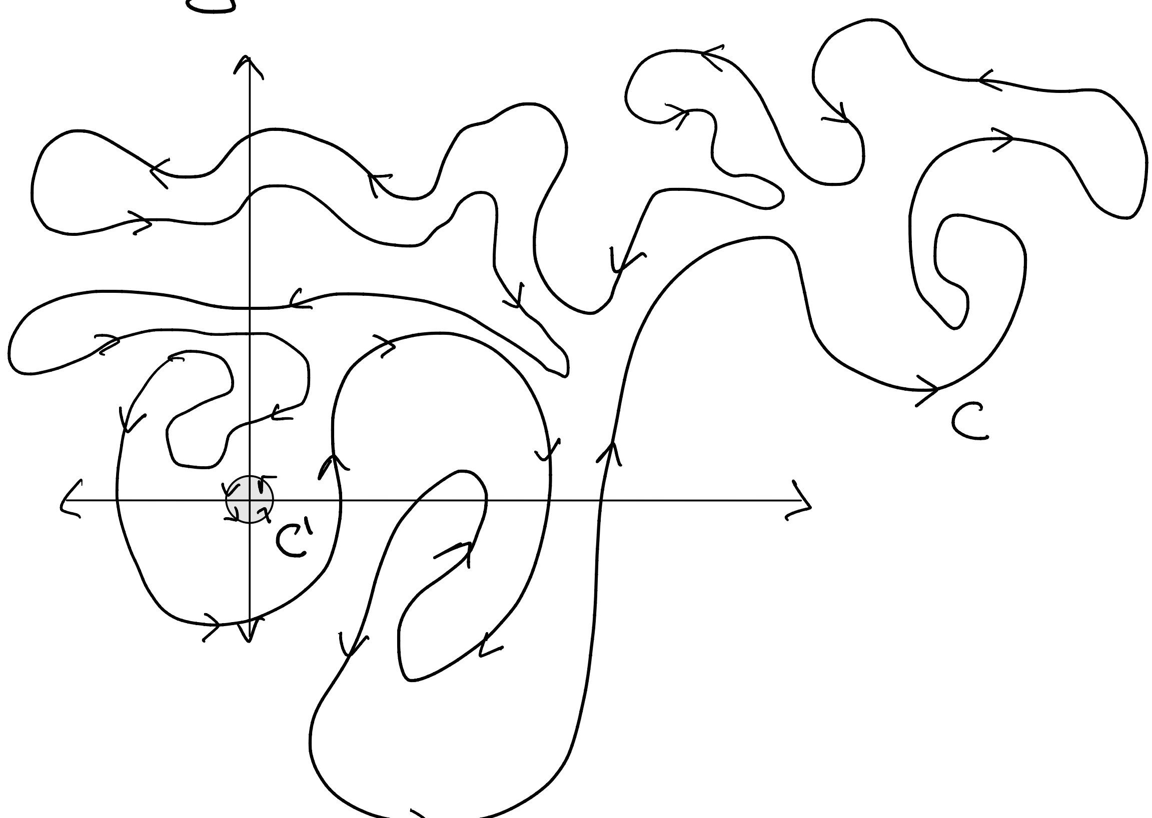
$$+ \frac{r\cos t}{r^2\cos^2 t + r^2\sin^2 t}(r\cos t)\,dt$$

$$= dt$$

(for any $r > 0$). Thus if C is any circle around the origin, we have

$$\int_C \omega = \int_0^{2\pi} dt = 2\pi.$$

Now suppose C is any simple closed curve which bounds a region including the origin.



There is a circle C' around the origin inside C, and a region S so $\partial S = C - C'$!

We then know

$$\int_C \omega = \int_{C'} \omega + \int_S d\omega = 2\pi + 0.$$

Definition. If C is a closed curve in $\mathbb{R}^2 - \{\vec{0}\}$, the <u>winding number</u> of C around $\vec{0}$ is given by

$$\frac{1}{2\pi}\int_C \frac{-y}{x^2+y^2}\,dx + \frac{x}{x^2+y^2}\,dy$$

Theorem. The winding number is an integer. If C and C' can be deformed continuously to one another, their winding numbers are equal.